Question order from yellow exam

1) Identify the expression equivalent to:

$$\frac{2x(x+2)^3(x^2+1)-(x+1)(x+2)^4(4x)}{x(x+2)^4}=\frac{2x(x+2)^3}{x(x+2)^4}\left(x^2+1-\left[(x+1)(x+2)(2)\right]\right)=$$

$$\frac{2}{(x+2)}\left(x^2+1-2\left(x^2+3x+2\right)\right)=\frac{2}{(x+2)}\left(-x^2-6x-3\right)=-\frac{2\left(x^2+6x+3\right)}{(x+3)}$$

First step (for me) was to factor out as much as I could (which also simplifies what's left of the numerator). Then I'm just multiplying out and combining similar terms in the 'left-over' expression in parentheses.

Notice also that there is an alternative way of doing this. Since the rational function is defined for $x = 1$, you can substitute 1 in for x and it simplifies to -20/3. This might be a daunting task for pencil and paper, but could be done with your calculators. It would be MUCH easier if you could substitute in a 0 for x – but the function is not defined at $x = 0$ (because of the x in the denominator). However … since we see that the x in the denominator eventually **cancels** then we know that there is a **hole** at $x = 0$ and **not** a vertical asymptote.

For those of you going on to MATH 241 – this could be an important point. These two expressions are algebraically equivalent **EXCEPT** in the particular case of $x = 0$ …..

The **correct answer is B**.

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2) Solve $x^3-23x+10=0$. The largest solution is in which of the following intervals?

It would be a mistake to just look at the largest **rational root**. However that is where we'll begin. Using the p/q strategy, we find that the only **possible** rational roots are $\pm1,\pm2,\pm5,\pm10$. You can proceed through the synthetic divisions a little more quickly if you realize that the '-23' is going to ruin the smaller tries, and the 10^3 is just going to be SO big, the other terms won't be able to cancel it. You find that a rational root is -5. The other factor is then x^2-5x+2.

The quadratic formula tells us that the other two solutions are $\frac{5\pm\sqrt{17}}{2}$. Since 17 is so close to 16, then we have x is **approximately** $\frac{5\pm4}{2}=\frac{9}{2},\frac{1}{2}$. So will be the largest solution and **E is correct**. This corresponds to the choice $(4,6]$.

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3) If -12 is a possible rational root of $a_nx^3-19x^2+37x+a_0=0$, which of the following are possible values of a_0 and a_n?

Notice the use of a_n as the leading coefficient instead of a_3 which would also be correct. This should suggest even more the use of the p/q thinking that solves this question easily.

-12 is a possible rational root, so 12 not only divides into a_0, but $12\cdot a_n$ also divides into a_0.

So, if $a_n = 1$, then possible values of a_0 are 12, 24, 36, 48, ….

If $a_n = 2$, then possible values of a_0 are 24, 48, 72, 96, …. And so on.

The **correct answer is B -** $a_0 = -36$, $a_n = 3$.

Note: This is still an easy version of this problem. The theorem said that the p/q could be formed from a **factor** of a_n … so a_n could really be **any** number. a_0 though would always have to be divisible by 12.

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4) Let $f(x)=\dfrac{1}{(x-a)^n}+b$ where a and b are positive real numbers and n is an even positive integer.

Determine which of the following statements are true.

I. The domain is $(-\infty,a)$.

II. The range is (b,∞).

III. The graph is increasing for (a,∞).

IV. $f(x)>b$ for all values of x for which the function is defined.

Note first that there are some **very** key words in the statement of this problem. "n is an even positive integer" tells us that the denominator is **always positive** (we are squaring something always) and the denominator **stays in the denominator** (the exponent can't be negative, so no fractions).

Let's talk about the 4 statements:

I.. ridiculous. Let $x = a + 1$. Clearly f is defined for this value of x, and this is outside the stated domain. So, I is false.

II. This is much more difficult. As we pointed out, the first term of f is always positive, and so the range might be a little smaller than II. How close can we get to b? If we look at our function again, we see a very close resemblance to $y=\dfrac{1}{x}$. The range of 1/x is every y-value except for 0. When we square the x we eliminate the negative values. So in our function we can get as close to 0 as we need to in that first term, and then add b to it, so our range is indeed the open interval (b,∞). **B is true**.

III. The 'right hand' side of our function (the branch to the right of the vertical asymptote) behaves just like the right hand branch of 1/x. And the right hand branch of $1/x$ **decreases** and does not increase. So **III is false**.

IV. Since II is true, **IV must be true** also.

So the **correct answer is C**.

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5) What is the x-intercept of $y=\sqrt[3]{2x+5}-3$?

The x-intercept occurs when -intercept occurs when y is 0. So

$0=\sqrt[3]{2x+5}-3\Rightarrow 3=\sqrt[3]{2x+5}\Rightarrow 3^3=(2x+5)\Rightarrow 22=2x\Rightarrow x=11$. So the x-intercept is $(11,0)$. The **correct answer is D**.

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6). Find values of A and B so that: $\dfrac{x+7}{x^2-x-2}=\dfrac{A}{x-2}+\dfrac{B}{x+1}$. What is $A+B$?

$\dfrac{A}{x-2}+\dfrac{B}{x+1}=\dfrac{A(x+1)+B(x-2)}{(x+1)(x-2)}$. So $A(x+1)+B(x-2)=x+7$. We can try the short cut approach.

When $x = 2$, we get $A(2+1)+B(0)=2+7\Rightarrow 3A=9\Rightarrow A=3$.

When $x = -1$, we get $A(0)+B(-3)=-1+7\Rightarrow -3B=6\Rightarrow B=-2$. And A+B is +1. **Correct is D**.

Alternative. $A(x+1)+B(x-2)=x+7\Rightarrow (A+B)x+(A-2B)=x+7$.

$A+B=1$ $\quad -A-B=-1$

So, $A-2B=7$ $\quad A-2B=7$ $\quad$ And, solving for A gives us $A = 3$.

$-3B=6\Rightarrow B=-2$

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7) Solve the following system of simultaneous equations: $\frac{x}{3}+\frac{y}{4}=\frac{3}{2}$ and $5y-7x=-11$.

What is the y–value of the solution?

$\frac{x}{3}+\frac{y}{4}=\frac{3}{2} \Rightarrow 4x+3y=18 \Rightarrow (eqn \bullet 5) \Rightarrow 20x+15y=90$

$-7x+5y=-11 \Rightarrow (eqn \bullet 3) \Rightarrow -21x+15y=-33$

So – taking opposite signs of equation 2 and adding - $41x=123 \Rightarrow x=3$. So y is then = 2.
The y value of the solution is 2 annd the **correct answer is C.**

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8) Solve the following system: $\begin{array}{l} x^2+y^2=25 \\ y=x^2-5 \end{array}$. Which of the following statements is **false**?

So we'll substitute to solve this one. $x^2=y+5$.

$y+5+y^2=25 \Rightarrow y^2+y-20=0 \Rightarrow (y+5)(y-4)=0 \Rightarrow y=-5,4$

$y=-5 \Rightarrow -5=x^2-5 \Rightarrow x^2=0 \Rightarrow x=0$

$y=4 \Rightarrow 4=x^2-5 \Rightarrow 9=x^2 \Rightarrow x=-3,+3$

We therefore have 3 solutions: $(-3,4),(3,4),(0,-5)$.

A) An x value of one of the solutions is -3. TRUE
B) The sum of the y values of all the solutions is 4+4+(-5)=3. TRUE
C) One of the y values is 4. TRUE
D) There are four solutions. FALSE (**and the correct answer**)

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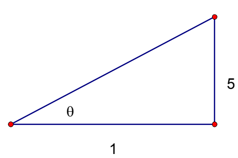
9) Given that $\cot\theta=\frac{1}{5}$ and the terminal side of θ is in quadrant III, determine $\sin\theta$.



The hypotenuse is $\sqrt{26}$. Then $\sin\theta=\frac{5}{\sqrt{26}}=\frac{5\sqrt{26}}{26}$. Now, theta is in the third quadrant. The sine function is negative in the third quadrant. So the **correct answer is** $-\frac{5\sqrt{26}}{26}$ **or E.**

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10) Determine which of the following is/are true.

I. $\frac{-5\pi}{12}=-150^\circ$. Well, $\frac{-5\pi}{12}=\frac{-5\pi}{12}\bullet\frac{180}{\pi}=-75^\circ$ **FALSE**

II. The radius of a sector with central angle of 45° and an area of 25π sq. ft. is 10 ft.

Let's check it – the area of a sector is $\theta r^2=\frac{45}{360}(100)=12.5\pi$. **FALSE.**

III. If the hypotenuse of an isosceles right triangle is 10 units, then the length of each leg of the triangle is $5\sqrt{2}$ units. We'll check with the Pythagorean Theorem.

$\left(5\sqrt{2}\right)^2+\left(5\sqrt{2}\right)^2=50+50=100=\left(\sqrt{100}\right)^2$. So the hypotenuse is 10! **TRUE**.

The **correct answer is E – III only.**

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11) Given that $\theta = \dfrac{67\pi}{6}$, determine which of the following is/are true.

We're going to figure out what quadrant that is in – for one, it's too big a number for me!

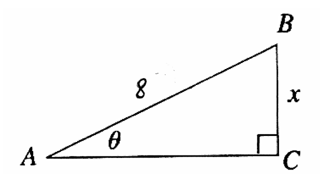
$\theta = \dfrac{67\pi}{6} = 11\pi + \dfrac{\pi}{6}$. Subtract off 2*pi 5 times, and I find the angle is equal to $\pi + \dfrac{\pi}{6} = \dfrac{7\pi}{6}$. That's in quadrant III (it's a little more than pi!)

I. The terminal side of θ is located in quadrant III. **TRUE**

II. $\cos\theta = \dfrac{-\sqrt{3}}{2}$. Check the triangle for the number part – cosine of 30 degrees is sqrt(3)/2. Cosine in the third quadrant is **negative**. **TRUE**

III. $\tan\theta = \dfrac{-\sqrt{3}}{3}$. This also **FALSE**. Tangents in quadrant III are positive.

So the **correct answer is D – I and II are true.**



12) Express $\tan\theta$ as a function of x.

$\tan\theta$ is the opposite side (x) divided by the adjacent side (but we need to find that). Using the Pythagorean Theorem, we see that third side is $\sqrt{64 - x^2}$. So $\tan\theta = \dfrac{x}{\sqrt{64 - x^2}}$. The **correct answer is A.**

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13) Which of the following is equivalent to $\dfrac{\sec\theta}{1 + \sec\theta}$?

$$\frac{\sec\theta}{1+\sec\theta} = \frac{\dfrac{1}{\cos\theta}}{\left(1 + \dfrac{1}{\cos\theta}\right)} \bullet \frac{\cos\theta}{\cos\theta} = \frac{1}{\cos\theta + 1}$$ and that is **answer C**.

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14) Find the period for the function $3\cos\left(2x + \dfrac{\pi}{4}\right)$.

The **period** of a function is the length of the x interval that corresponds to one full cycle of the function. For $y = \cos(x)$, the period is 2π. So what is the x range that leads to the 2π length interval that *cos* would see?

Let's begin with $x = 0$. Then *cos* sees $2 \bullet 0 + \dfrac{\pi}{4} = \dfrac{\pi}{4}$. So we'll add one *cos* period to that (2π) and see what value of x corresponds to $2\pi + \dfrac{\pi}{4}$. $2x + \dfrac{\pi}{4} = 2\pi + \dfrac{\pi}{4} \Rightarrow 2x = 2\pi \Rightarrow x = \pi$. So the period is Pi. **Answer C**

Note that all we have done is to solve the inequality in the book: $0 \le 2x + \dfrac{\pi}{4} \le 2\pi$ to determine the fundamental
period … I just did it with easier arithmetic.

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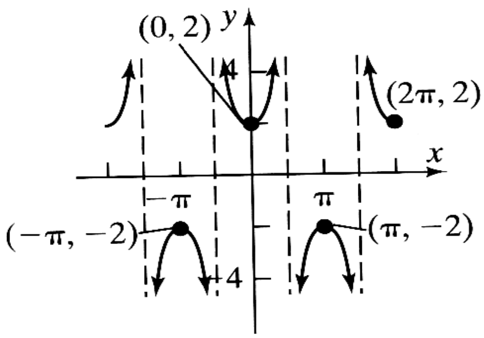
15) Consider the graph below.



Which of the following functions corresponds to this graph?
I hope you recognize this curve as a secant or cosecant. But which? Notice that the y-intercept is 2. Since there **IS** a y-intercept, we conclude this graph should be related to $1/\cos(x)$, which is the secant. Sec(x) however has a y-intercept of 1. This needs to be twice that, so we now consider the function to be $2\sec(x)$.

Let's see if the period or horizontal 'component' of the graph match. We know that cosine has zeros at $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Since our graph has vertical asymptotes at those two values of x, we know our graph is $y = 2\sec(x)$. The **correct answer is D.**

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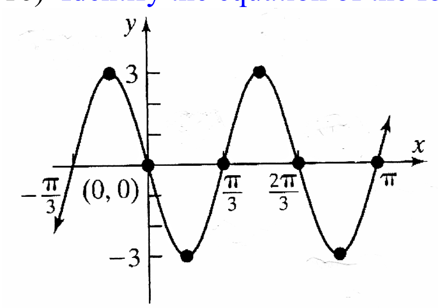
16) Identify the equation of the following graph:



We look over the choices and see we need to decide on the frequency and the phase shift. It looks like there is a complete cycle from $x = 0$ to $x = \frac{2\pi}{3}$. So we can fit *3* of them between 0 and 2π. That's our frequency …. But notice that as we consider the interval to the right of $x = 0$, the graph is **decreasing**. So, we need to consider a phase shift. Look at the period that starts at $\frac{\pi}{3}$ - that looks like a regular sin curve. We can therefore translate our graph horizontally (to the right) by $\frac{\pi}{3}$.

Putting all this together our curve is $3\sin\left(3\left(x - \frac{\pi}{3}\right)\right) = 3\sin(3x - \pi)$

Our **correct answer is A.**

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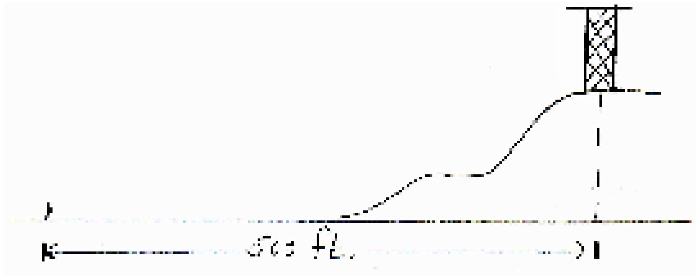
17) A communications tower is located on the top of a hill. The angle of elevation from a point on level ground 500 feet from the hill to the top of the tower is 57°. The angle of elevation form the same point to the bottom of the tower is 48°. How tall is the tower?



We have two clues. For the top of the tower let's call the height h_T we have $\tan 57^\circ = h_T / 500 \Rightarrow h_T = 500 \tan 57^\circ = 769.93$. For the bottom of the tower let's call the height h_B we have $\tan 48^\circ = h_B / 500 \Rightarrow h_B = 500 \tan 48^\circ = 555.31$. So the height of the tower must be 769.93 - 555.31 which is about 214.6 feet. So the **correct answer was A**.

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18) Determine the exact value of: $\sin^{-1}\left(\cos\left(\frac{2\pi}{3}\right)\right)$.

$\cos\left(\frac{2\pi}{3}\right) = -\cos\left(\frac{\pi}{3}\right) = -\frac{1}{2}$. $\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$.

The **correct answer is E.**

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19) Let $f(x) = \dfrac{2x^2 + 7x - 15}{2x^2 - 8x}$.

Vertical Asymptotes:

$$2x^2 - 8x = 0 \Rightarrow 2x(x-4) = 0$$

So the **vertical asymptotes are $x = 0$, and $x = 4$.**

Horizontal Asymptotes:

The end behavior of polynomials is the same as the end behavior of the high degree terms. So just look at $\dfrac{2x^2}{2x^2} = 1$. The **horizontal asymptote is a line: $y = 1$**

Zeros:

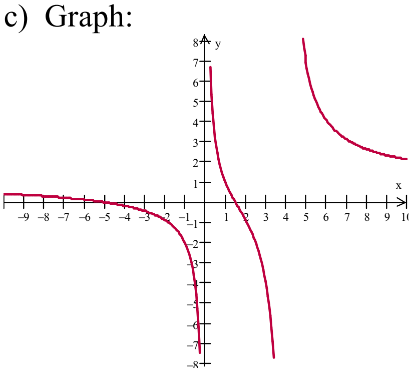
Set the numerator equal to zero and solve.

$$2x^2 + 7x - 15 = 0 \Rightarrow (2x-3)(x+5) = 0 \Rightarrow x = -5, \frac{3}{2}$$

b) Does f intersect the horizontal asymptote?

$$\frac{2x^2 + 7x - 15}{2x^2 - 8x} = 1 \Rightarrow 2x^2 + 7x - 15 = 2x^2 - 8x \Rightarrow 15x = 15 \Rightarrow x = 1$$

c) Graph:



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20) For the equation $y = 3\sin\left(x + \frac{\pi}{4}\right)$, find the following:

a) The amplitude: 3
b) The period: 2π
c) Phase shift: $-\frac{\pi}{4}$. Same idea as horizontal translation
d) The 5 guidepoints: $\frac{-\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$
e) Check on your Ti8* from high school or ask in class…. (I'm tired!)

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21) Find all solutions to the equation $2\sin^2\theta + 1 = 3\sin\theta$ on the interval $[0, 2\pi)$. Give exact values of θ, not decimal approximations.

$2\sin^2\theta + 1 = 3\sin\theta \Rightarrow 2\sin^2\theta - 3\sin\theta + 1 = 0$

Notice the resemblance to $2x^2 - 3x + 1 = 0 \Rightarrow (2x-1)(x-1) = 0 \Rightarrow x = \frac{1}{2}, 1$

But 'x' in the quadratic stood for sin(theta), so either

$\sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$ or $\sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$. Three possible solutions.

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22) Verify the following identity: $\frac{\tan^2 x}{\sec x + 1} = \frac{1 - \cos x}{\cos x}$.

$$\frac{\tan^2 x}{\sec x + 1} = \frac{\sin^2 x / \cos^2 x}{\left[(1/\cos x) + 1\right]} \bullet \frac{\cos^2 x}{\cos^2 x} = \frac{\sin^2 x}{\cos x + \cos^2 x} = \frac{1 - \cos^2 x}{\cos x(1 + \cos x)} = \frac{(1 - \cos x)(1 + \cos x)}{\cos x(1 + \cos x)} = \frac{(1 - \cos x)}{\cos x}$$

OR

$$\frac{\tan^2 x}{\sec x + 1} = \frac{\sec^2 x - 1}{\sec x + 1} = \frac{(\sec x - 1)(\sec x + 1)}{\sec x + 1} = \sec x - 1 = \frac{1}{\cos x} - 1 = \frac{1}{\cos x} - \frac{\cos x}{\cos x} = \frac{1 - \cos x}{\cos x}$$

There are several other ways to get to the same place ….

Do let me know if this helped you …..